Q.2 a. Show by using Mathematical Induction that

$$\sum_{i=1}^{n} i^2 = \frac{n\cdot(n+1)\cdot(2\cdot n+1)}{6}$$

Answer:

Solution: BASIS: At $n = 1$, the LHS is $\sum_{i=1}^{1} i^2 = 1^2 = 1$. Likewise, the RHS is $\frac{1\cdot 2\cdot(2\cdot 1+1)}{6} = 1$. Since the LHS and RHS are equal, the basis is proven.

INDUCTIVE STEP: Assume that for some $k \geq 1$,

$$\sum_{i=1}^{k} i^2 = \frac{k\cdot(k+1)\cdot(2\cdot k+1)}{6}.$$

$$\begin{aligned}
\sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^{k} i^2 + (k+1)^2 \\
&= \frac{k\cdot(k+1)\cdot(2\cdot k+1)}{6} + (k+1)^2, \text{ using the inductive hypothesis} \\
&= \frac{(k+1)}{6}\cdot[k\cdot(2\cdot k+1) + 6\cdot(k+1)] \\
&= \frac{(k+1)}{6}\cdot[2\cdot k^2 + k + 6\cdot k + 6] \\
&= \frac{(k+1)}{6}\cdot[2\cdot k^2 + 7\cdot k + 6] \\
&= \frac{(k+1)}{6}\cdot[2\cdot k^2 + 4\cdot k + 3\cdot k + 6] \\
&= \frac{(k+1)}{6}\cdot[2\cdot k(k+2) + 3\cdot(k+2)] \\
&= \frac{(k+1)\cdot(k+2)\cdot(2\cdot k+3)}{6}
\end{aligned}$$

b. Define language. Let ∑= {0; 1} denote an alphabet. Enumerate five elements of the following languages:

(i) Even binary numbers,

Answer: Even binary numbers: {0; 10; 100; 110; 1000}

(ii) The number of zeros is not equal to the number of ones in a binary string.

Answer: The number of zeros is not equal to the number of ones in a binary string: {0; 1; 100; 110; 001}

(iii) The number of zeros is exactly one greater than the number of ones.

Answer: The number of zeros is exactly one greater than the number of ones: {0; 100; 001; 010; 00011}

Q.3 a. Construct DFA to accept all possible strings of 0's and 1's which does not contain 011 as a substring.

Answer:

State	δ	
	0	1
→ * q_0	q_1	q_0
* q_1	q_2	q_2
* q_2	q_1	q_3
q_3	q_3	q_3

b. Obtain DFA from the following NFA defined by transition table given below:

State	δ		
	0	**1**	**2**
$\rightarrow q_0$	$\{q_0, q_1, q_2\}$	$\{q_1, q_2\}$	$\{q_2\}$
q_1	Φ	$\{q_1, q_2\}$	$\{q_2\}$
$*q_2$	Φ	Φ	$\{q_2\}$

Answer:

The required DFA is M = (Q, Σ, δ, q0, F), where Q = {{q0}, {q0, q1, q2}, {q1, q2}, {q2}}, Σ = {0, 1, 2}, F = {q0, q1, q2} and transition function is defined by the transition table given below.

State	δ		
	0	1	2
→{q0}	{q0, q1, q2}	{q1, q2}	{q2}
*{q0, q1, q2}	{q0, q1, q2}	{q1, q2}	{q2}
{q1, q2}	Φ	{q1, q2}	{q2}
{q2}	Φ	Φ	{q2}

c. Prove that a language L is accepted by some ε-NFA if and only if L is accepted by some DFA.

Answer: Page Number 75 of Text Book.

Q.4 a. Prove that if L and M are regular languages, then so is $L \cap M$

Answer: Page Number 126 of Text book.

b. Define Context Free Grammar. Give Context Free Grammar that generates the following Languages:

(i) $L = \{w \in \{0, 1\}^* \mid w$ contains more 1's than 0's$\}$
(ii) $L = \{a^i b^j c^k \mid i, j,k \geq 0$ and either $i = j$ or $j = k\}$
(iii) $L = \{a^n u \mid u \in \{a, b\}^*$ and $|u| = n, n \geq 0\}$

Answer: Page Number 159 of Text Book.

(i) $L = \{w \in \{0, 1\}^* \mid w$ contains more 1's than 0's$\}$
$S \rightarrow 0S1 \mid 1S0 \mid S01 \mid S10 \mid 01S \mid 10S \mid 1S \mid 1$

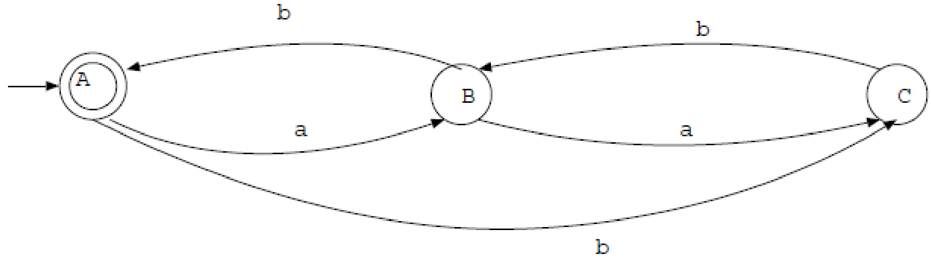
(ii) L = { aibjck |i,j,k≥0 and either i = j or j = k}

$S \rightarrow AC \mid B$
$A \rightarrow aAb \mid e$
$C \rightarrow cC \mid e$
$B \rightarrow aB \mid D$
$D \rightarrow bDc \mid \varepsilon$

(iii) $L=\{a^n u | u \in \{a, b\}^*$ and $|u|=n, n \geq 0\}$

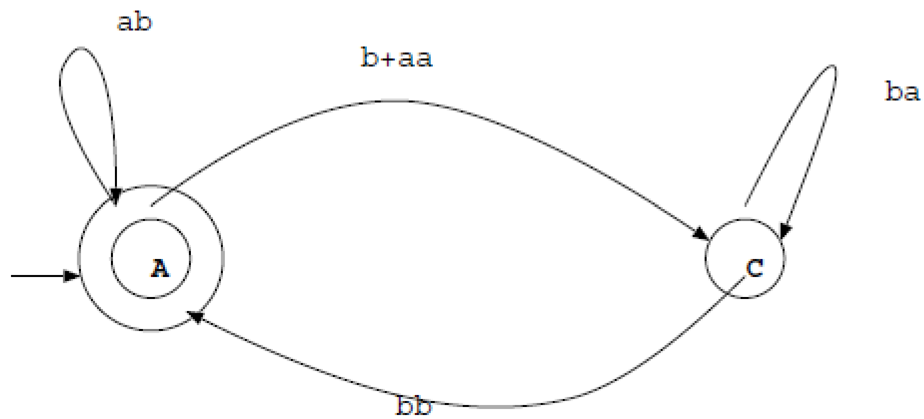
$S \rightarrow aSa \mid aSb \mid e$

Q.5 a. Write a regular expression for the language represented by the Finite Automata given in the below figure:

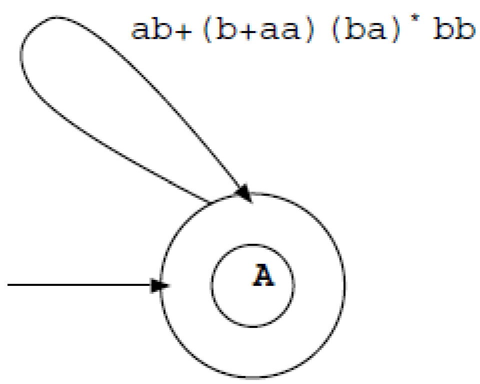


Answer:

After eliminating state B



After eliminating state C



The final RE is $(ab + (b + aa) \cdot (ba)^* \cdot bb)^*$

Q.6 a. Design a PDA to accept the language $L = \{a^i b^j c^k : i + j = k; i \geq 0, j \geq 0\}$

Answer:

PDA $M = (\{q_0, q_1, q_2, q_3\}, \{a, b, c\}, \{a, b, Z_0\}, \delta, q_0, Z_0, \{q_3\})$, where δ is defined by following rules:

$\delta(q_0, a, Z_0) = \{(q_0, aZ_0)\}$

$\delta(q_0, a, a) = \{(q_0, aa)\}$

$\delta(q_0, b, a) = \{(q_1, ba)\}$

$\delta(q_0, b, Z_0) = \{(q_1, bZ_0)\}$

$\delta(q_0, c, a) = \{(q_2, \varepsilon)\}$

$\delta(q_1, b, b) = \{(q_1, bb)\}$

$\delta(q_1, c, b) = \{(q_2, \varepsilon)\}$

$\delta(q_2, c, b) = \{(q_2, \varepsilon)\}$
$\delta(q_2, c, a) = \{(q_2, \varepsilon)\}$
$\delta(q_2, \varepsilon, Z_0) = \{(q_3, \varepsilon)\}$
$\delta(q_0, \varepsilon, Z_0) = \{(q_3, \varepsilon)\}$
$\delta(q, x, Y) = \phi$ for all other possibilities

b. Consider the grammar G = (V, T, S, P), with productions defined by: S → aSbS | bSaS | ε. Is G ambiguous? If so, prove and also provide the unambiguous grammar for the same language.

Answer:
G is ambiguous, since the string w = abab has two distinct leftmost derivations:

(i) S ⇒ aSbS ⇒ abSaSbS ⇒ abaSbS ⇒ ababS ⇒ abab, and
(ii) S ⇒ aSbS ⇒ abS ⇒ abaSbS ⇒ ababS ⇒ abab.

L(G) is the language of strings over {a, b}, in which the number of as is equal to the number of bs. An unambiguous grammar for this language is given by: G′ = (V, T, S, P), where V = {S}, T = {a, b}, S = S, and P = {S → aSb | bSa | SS | ε}.

c. What is the relationship between Deterministic Push Down Automata, regular Languages and Context Free Languages?

Answer:
The languages accepted by DPDA's by final state properly include the regular language, but are properly included in the CFL

Q.7 a. Convert the following simplified grammar to CNF
G = ({S, A, B}, {a, b}, {S → ASB | AB, A → aAS | a | aA, B → SbS | bb | Sb | bS | b | aAS | a | aA }, S)

Answer:
Step 1: Add productions of the form A → BC and A → a to P′. So P′ = {S → AB, A → a, B → b | a}

Step 2: Eliminate terminals from RHS of the other productions
A → aAS to A → C_aAS and C_a → a
A → aA to A → C_aA
B → SbS to B → SC_bS and C_b → b
B → bb to B → C_bC_b
B → bS to B → C_bS
B → Sb to B → SC_b
B → aAS to B → C_aAS

$B \rightarrow aA$ to $B \rightarrow C_aA$

Add productions of the form $A \rightarrow BC$ to P′. So P′ = {$S \rightarrow AB$, $A \rightarrow a \mid C_aA$, $B \rightarrow C_bC_b \mid C_bS \mid SC_b \mid C_aA \mid b \mid a$, $C_a \rightarrow a$, $C_b \rightarrow b$ }

Step 3: Reduce the RHS of the productions with more than 2 variables to the form of $A \rightarrow BC$.

$A \rightarrow C_aAS$ to $A \rightarrow C_aC_1$ and $C_1 \rightarrow AS$
$B \rightarrow SC_bS$ to $B \rightarrow SC_2$ and $C_2 \rightarrow C_bS$
$B \rightarrow C_aAS$ to $B \rightarrow C_a C_3$ and $C_3 \rightarrow AS$

Adding these productions to P′ = { $S \rightarrow AB$, $A \rightarrow a \mid C_aA \mid C_aC_1$, $B \rightarrow C_bC_b \mid C_bS \mid SC_b \mid C_aA \mid SC_2 \mid C_aC_3 \mid b \mid a$, $C_a \rightarrow a$, $C_b \rightarrow b$, $C_1 \rightarrow AS$, $C_2 \rightarrow C_bS$, $C_3 \rightarrow AS$}

The grammar in CNF form G′ = (V′, {a, b}, P′, S)
V′ = {S, A, B, C_a, C_b, C_1, C_2, C_3}
P′ = { $S \rightarrow AB$, $A \rightarrow a \mid C_aA \mid C_aC_1$, $B \rightarrow C_bC_b \mid C_bS \mid SC_b \mid C_aA \mid SC_2 \mid C_aC_3 \mid b \mid a$, $C_a \rightarrow a$, $C_b \rightarrow b$, $C_1 \rightarrow AS$, $C_2 \rightarrow C_bS$, $C_3 \rightarrow AS$}

b. State Pumping Lemma for Context Free Language. Show that the language, L={$0^i1^j2^i3^j$ | i ≥1, j≥1} is not a Context Free Language.

Answer:

Page Number 258 of Text Book.

- Assume L is CFL
- Pick $z = uvwxy = 0^n1^n2^n3^n$,$|vwx| \leq n$ & $vx \neq \varepsilon$
- vwx can consist of a substring of one symbol
  - uwy has n of 3 different symbols and fewer than n of 4th symbol. Then uwy is not in L
- vwx can consist of 2 adjacent symbols, say 1 & 2
  - uwy is missing some 1's or 2's and uwy is not in L
- L is not CFL

Q.8 a. Design a Turing Machine to accept the language L = {w ∈ {a, b}*| w is a palindrome}. Give traces of the machine for the strings "baab" and "ababa"

Answer:

Design of TM

state	A	b	X	Y	B
→ q0	(q1, X, R)	(q2, Y, R)	(q6, X, R)	(q6, Y, R)	(q6, B, R)
q1	(q1, a, R)	(q1, b, R)	(q3, X, L)	(q3, Y, L)	(q3, B, L)
q2	(q2, a, R)	(q2, b, R)	(q4, X, L)	(q4, Y, L)	(q4, B, L)
q3	(q5, X, L)	-	(q6, X, R)	(q6, Y, R)	-
q4	-	(q5, Y, L)	(q6, X, R)	(q6, Y, R)	-
q5	(q5, a, L)	(q5, b, L)	(q0, X, R)	(q0, Y, R)	-
*q6	-	-	-	-	-

Trace of “baab”

q0baab ├── Yq2aab ├── Yaq2ab ├── Yaaq2b ├──Yaabq2 ├──Yaaq4b ├── Yaq5aY ├── Yq5aaY ├── q5YaaY ├──Yq0aaY ├── YXq1aY ├── YXaq1Y ├──YXq3aY ├──Yq5XXY ├──YXq0XY ├── YXXq6Y (accept)

Similarly for the traces of “ababa”

b. Prove that every language accepted by a multitape Turing Machine is recursively Enumerable.

Answer: Page Number 316 of Text Book.

Q.9 Define the following languages with diagram:

(iii) Recursive

Answer: Page Number 349 of Text Book

TEXT BOOK

Introduction to Automata Theory, Languages & Computation, John E Hopcraft, Rajeev Motwani, Jeffery D. Ullman, Pearson Education, 3rd Edition, 2006